LESSONS 1-3: Number System & Number Sense

This covers:

- Introduction to real numbers
- Laws of exponents with integral powers
- Properties of real numbers
- Rationalization of real numbers

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Introduction to real numbers (Lesson 1)

Definition: Real numbers (R) are the collection of all rational and irrational numbers. Each real number can be represented by a unique point on the number line.

1. Rational and irrational numbers

i. What are rational numbers?

Definition: Rational numbers (Q) are numbers that can be written as simple fractions. They include all positive and negative numbers that can be written in the form $\frac{p}{q}$, where *p* and *q* are integers, and *q* does not equal 0.

Note that an infinite number of equivalent fractions can be used to represent the same rational number. For example: $\frac{2}{4} = \frac{4}{8} = \frac{10}{20} = \frac{1}{2}$. We can use $\frac{1}{2}$ to represent all of these fractions.

Examples of rational numbers:

- ½ is a rational number. Here, *p* = 1 and *q* = 2.
- 0 is a rational number. We can write it in the form $\frac{p}{q}$ like this: $\frac{0}{5}$ (This is one example of how we can write zero as a fraction. There are an infinite number of others.) Here, *p* = 0 and *q* = 5.
- −14 is a rational number. We can write it in the form $\frac{p}{q}$ like this: $\frac{-14}{1}$. Here, *p* = −14 and *q* = 1.

All natural numbers, whole numbers and integers are included in the rational numbers.

Definitions (recap):

- **Natural Numbers** (N) are whole counting numbers like 1, 2 and 3. The smallest natural number is 1, and the numbers continue increasing infinitely from 1.
- **Whole Numbers** (N_0) include all the natural numbers and also the number 0.
- **Integers** (Z) are all the positive and negative whole numbers, as well as the number 0.

ii. Recurring and terminating decimals

How do we know if a decimal number is rational? Decimal numbers will fit into one of two main categories:

i. Terminating decimals: This means that the decimal expansion ends after a finite number of steps.
E.g.: $\frac{1}{8} = 0.125$
All terminating decimals are **rational**, because they can be written in the form $\frac{p}{q}$, where *p* and *q* are integers, and *q* does not equal 0.

ii. Non-terminating decimals: This means that the decimal expansion goes on forever – you never reach the final digit.
Non-terminating decimals can be further categorised as being either *recurring* or *non-recurring*:

- A non-terminating decimal is *recurring* if there is a repeating digit or block of digits after the decimal point. The repeated digit or digits will continue repeating forever.

We use a line on top of the repeating digit or digits to tell us that they are repeating.
E.g.: $\frac{1}{3} = \mathbf{0.333333333\ldots = 0.\bar{3}}$
E.g.: $3.5727272\ldots = 3.5\bar{7}\bar{2}$
All decimals that are non-terminating and recurring, are **rational**.

- A non-terminating decimal is *non-recurring* if there is not a repeated digit or block of digits after the decimal point.
All decimals that are non-terminating and non-recurring, are **irrational**.

Definitions:

- A **terminating decimal** is a number with a decimal expansion that ends after a finite number of steps.
- A **non-terminating decimal** is a number with a decimal expansion that goes on forever – you never reach the final digit.

iii. Decimal expansions

To write out a decimal expansion for a given fraction, we can use long division.
Note that fractions are rational numbers, so if you keep dividing you will either come to the end of the expansion (this means the number is a terminating decimal) or you will see that a digit or group of digits starts repeating (this means that the number is a non-terminating, recurring decimal).

Worked Example
Write the following fraction in decimal form and say what kind of decimal expansion it has: $\frac{1}{8}$.

Worked Solution

$$8\overline{)1}\quad 0.125$$

0
10
8
20
16
40
40
0

This is a terminating decimal.

Worked Example
Write the following fraction in decimal form and say what kind of decimal expansion it has: $\frac{1}{3}$.

Worked Solution

$$3\overline{)1}\quad 0.333\ldots$$

0
10
9

10
9
10
9
1

This is a non-terminating, recurring decimal.

iv. What are irrational numbers?

Definition: Irrational numbers are the collection of positive and negative numbers that cannot be written in the form $\frac{p}{q}$, where *p* and *q* are integers, and *q* does not equal 0. Irrational numbers include non-terminating non-recurring decimals.

There are infinitely many irrational numbers.

Examples of irrational numbers:

$\pi, \sqrt{5}, \sqrt{3}, \sqrt{2}$

Laws of exponents (Lesson 2)

1. Revision of exponent laws

You have learnt the laws of exponents in previous classes:

1. $a^m . a^n = a^{m+n}$
2. $(a^m)^n = a^{m.n}$
3. $\frac{a^m}{a^n} = a^{m-n} \qquad (m > n)$
4. $a^m b^m = (ab)^m$
5. $a^{\frac{m}{n}} = (\sqrt[n]{a})^m = \sqrt[n]{a^m}$

These laws can be extended to include the following:

6. $a^0 = 1$
7. $a^{-1} = \frac{1}{a}$

Properties of real numbers(Lesson 3)

1. The number line

i. Rational numbers

All rational numbers can be represented on a number line.

It is easy to find the location of ½, 0 or –14 on a number line. But what about a rational number like 0.1255? For a number such as this, we can find its position on a number line using the *process of successive magnification*. This means that we find the position of the number by constantly 'zooming in' to the number line.

Definition: Successive Magnification is a method we can use to locate rational decimal numbers on the number line.

Here is how this works:

- Find the two whole numbers on the number line that the decimal lies in between.
- Divide the distance between the two whole numbers into tenths.
- Find the two tenths that the number lies in between.
- Divide the distance between the two tenths into hundredths.
- Continue in this way until you have found the exact position of the rational number on the number line.

Worked Example

Show the number 2.445 on the number line.

Worked Solution

We start with the given number: **2.445**.

We know that this number lies somewhere in between 2 and 3. Let's divide the space between 2 and 3 into tenths:

2 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 2.9 3

Where do you think you will find 2.445 now? We know that 2.445 should lie somewhere in between 2.4 and 2.5:

2 2.1 2.2 2.3 2.4 2.5 2.6 2.7 2.8 2.9 3

↑

But we still don't know where exactly it should lie, so let's divide this space up into 10 units further:

2.40 2.41 2.42 2.43 2.44 2.45 2.46 2.47 2.48 2.49 2.50

Now we know that 2.445 lies exactly half-way in between 2.44 and 2.45:

2.40 2.41 2.42 2.43 2.44 2.45 2.46 2.47 2.48 2.49 2.50

↑

And this means that you can locate its exact position on the number line:

2.440 2.441 2.442 2.443 2.444 2.445 2.446 2.447 2.448 2.449 2.450

↑

If you are asked to find the position of a non-terminating, recurring decimal on the number line you will also be able to use the method shown above. However, you will be told the degree of accuracy with which you will need to locate the decimal on the number line. Otherwise, you can go on magnifying the number line forever!

ii. Irrational numbers

All irrational numbers can be represented on a number line. We can use the process of successive magnification to find the position of an irrational number on the number line up to a certain accuracy.

Definition: Successive Magnification is a method we can use to locate rational decimal numbers on the number line.

For irrational numbers that are the square root of a whole number, we can go one step further by finding their exact position using the Theorem of Pythagoras.

Here is how this works:

- Find two numbers that, when squared, will add up to the number inside the square root sign.
- These numbers will form the two sides of a right-angled triangle with the square root as the hypotenuse.
- Measure out the distances of the two perpendicular sides of the triangle and construct the triangle on the number line.
- Use a pair of compasses and move the arms so that they are the exact distance of the hypotenuse of the right-angled triangle.
- Mark this exact distance on the number line.

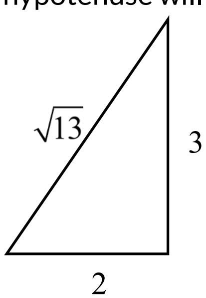
Worked Example

Find the exact position of $\sqrt{13}$on the number line.

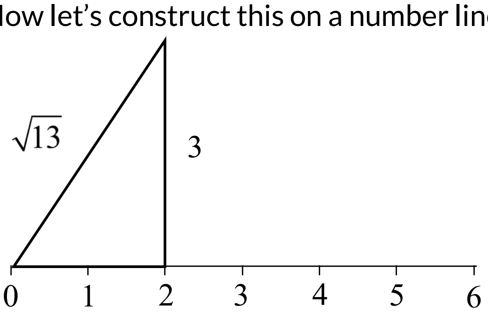
Worked Solution

$2^2 + 3^2 = 13$

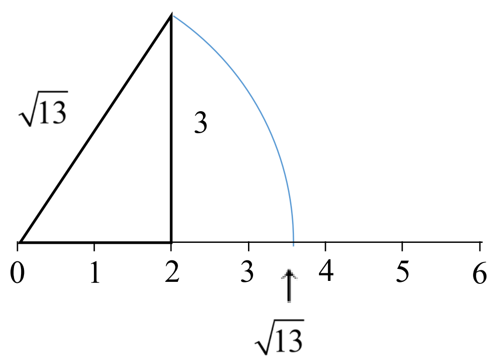
So if we draw a right-angled triangle with one side equal to 2 units and the other side equal to 3 units, the hypotenuse will be equal to $\sqrt{13}$ units:



Now let's construct this on a number line:



Using a pair of compasses, measure the length of the hypotenuse and set this as the radius. Draw an arc using this radius, which joins the tip of the triangle with the number line:



Important: All the rational and irrational numbers together make up the group called the real numbers. Every real number is represented by a unique point on the number line and conversely, every point on the number line represents a unique real number.

2. Rules for real numbers

There are certain rules that stay true when you are working with rational and irrational numbers. You can use these rules to test whether a certain number is rational or irrational.

The rules are:
- If we add, subtract, multiply or divide (except by zero) any two rational numbers, the result is always a **rational** number.
- The sum or difference of a rational number and an irrational number is always **irrational**.
- The product or quotient of a non-zero rational number and an irrational number is **irrational**.
- If we add, subtract, multiply or divide two irrationals, the result may be either **rational** or **irrational**.

Worked Example
Is the number $2\sqrt{13}$ rational or irrational?

Worked Solution:
We know that 2 is a rational number, and $\sqrt{13}$is irrational.
We also know that the product or quotient of a non-zero rational number and an irrational number is **irrational**.
Therefore $2\sqrt{13}$is irrational.

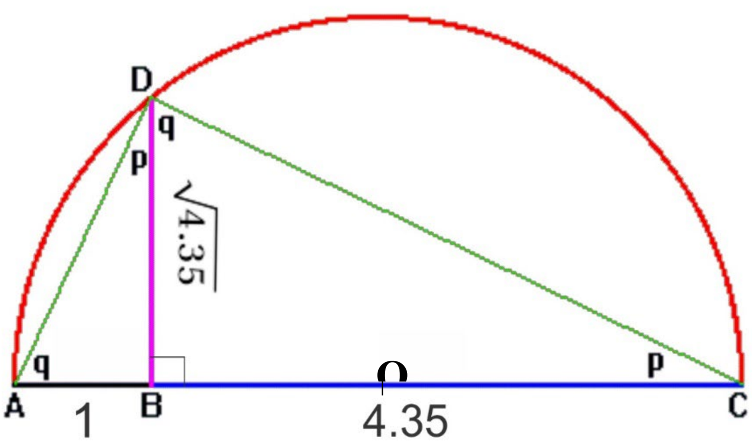
3. Square roots

Does the square root of every positive real number give us an answer that is also a real number? The answer is yes, and we can prove this using Geometry.

To find any length $\sqrt{\boldsymbol{x}}$, we can do the following:

i. Measure a distance equal to *x* (the number inside the square root sign) on a line. Mark the beginning of this distance as B and the end as C.
ii. Measure a distance of 1 unit further along the line from B and mark this point as A.
iii. Find the point on the line that is exactly halfway between A and C, and call this O.
ii. Now take a pair of compasses and set their width to be the same as the distance from A to O. Draw a semicircle that starts at point A and ends at point C.
iii. Now draw a line that is perpendicular to AC, starting at point B and joining the semicircle at a point that you can call D.
iv. The length of the line BD will be equal to $\sqrt{\boldsymbol{x}}$.

Here is the example we used in the video lesson, using the length $\sqrt{\mathbf{4.35}}$:



You can use this method to find the square root of any positive real number. The square root of any positive real number will give you a real solution.

4. The *n*th root

Every positive real number has a single positive *n*th root. The *n*th root is the number that is used *n* times in a multiplication to give you the number that you started with.

For example, for the square root of a number *n* = 2. A number is multiplied by itself twice when it is squared; to find the original number we take the second root (also known as the square root):
$2^2 = 4$; $\sqrt{4} = 2$

If *n* = 3, we follow the same rule: A number is multiplied by itself three times when it is cubed; to find the original number we take the third root (also known as the cube root):
$2^3 = 8$; $\sqrt{8} = 2$

So the letter *n* in the *n*th root is a placeholder that represents any positive real number.

Definition: The *n*th root of a real number is a number that can be multiplied by itself *n* times, to get to a given number. The *n* is a placeholder that represents any positive real number.

5. Rationalisation

i. A square root

Consider the following numbers: $\frac{1}{\sqrt{3}}$; $\frac{1}{1+2\sqrt{3}}$; $\frac{1}{\sqrt{2}+\sqrt{3}}$
Can you work out where they will fall on the number line?
It is difficult to work with fractions that have denominators that are irrational. To solve this problem, we can rationalise the denominator, which means to find an equivalent fraction that has a rational denominator.

Definition: To rationalise the denominator of a fraction with an irrational denominator means to find an equivalent fraction that has a rational denominator.

Worked Example
Rationalise the denominator of $\frac{1}{\sqrt{3}}$.

Worked Solution
We need to find a number that we can multiply the denominator by, to make it rational.
You know that we can multiply a square root by itself to get rid of the square root sign: $\sqrt{a} \times \sqrt{a} = a$
But you do not want to change the original fraction, so make sure that you multiply the numerator by the same number:
$\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{\sqrt{3}}{3}$
This is an easier number to work with. If you are asked to find it on the number line, you can say that $\frac{\sqrt{3}}{3}$ can be found exactly a third of the way between 1 and $\sqrt{3}$.

ii. More than 1 term

If there is more than one term in the denominator, you will need to approach the rationalisation a bit differently.

We still need to get rid of the square root by multiplying it by itself.
Here you can use the identity: $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$

Worked Example
Rationalise the denominator of $\frac{1}{1+2\sqrt{3}}$.

Worked Solution
You know that $(a + \sqrt{b})(a - \sqrt{b}) = a^2 - b$
So:

$$(1+2\sqrt{3})(1-2\sqrt{3}) = (1)^2 - (2\sqrt{3})^2$$
$$= 1 - 12$$
$$= -11$$

We can use this to get rid of the square root in the denominator:

$$\frac{1}{1+2\sqrt{3}} \times \frac{1-2\sqrt{3}}{1-2\sqrt{3}} = \frac{1-2\sqrt{3}}{1-12}$$
$$= \frac{1-2\sqrt{3}}{-11}$$
$$= -11 + \frac{2\sqrt{3}}{11}$$

iii. 2 square root terms

For fractions with two square roots in the denominator, you can use the same theory as we used in the previous example. The identity is as follows: $(\sqrt{a}+\sqrt{b})(\sqrt{a}-\sqrt{b}) = a - b$

Worked Example

Rationalise the denominator of $\frac{1}{\sqrt{2}+\sqrt{3}}$.

Worked Solution

Remember to multiply both the numerator and denominator by the same number, and simplify your answer.

$$\frac{1}{\sqrt{2}+\sqrt{3}} \times \frac{\sqrt{2}-\sqrt{3}}{\sqrt{2}-\sqrt{3}} = \frac{\sqrt{2}-\sqrt{3}}{2-3}$$
$$= \frac{\sqrt{2}-\sqrt{3}}{-1}$$
$$= -\sqrt{2} + \sqrt{3}$$