LESSONS 14-19: CIRCLES

This covers:

1. Basic terms
2. Properties related to chord and angle of circle
3. Perpendicular from the centre to a chord
4. Circle through three non-collinear points
5. Perpendicular distance between the chord and the circle
6. Angle subtended by an arc of a circle
7. Concyclic points
8. Cyclic quadrilaterals

Notes Breakdown	
Circles (Unit 6)	
Basic Terms	
1. Basic terms used to describe circles	
Parts of a circle	Parts of a circle
Lines and circles	Lines and circles
Subtended angles	Subtended angles
More than one circle	More than one circle
Circles theorems	
1. Properties related to chord and angle of circle	Chords & angles
Equal chords of a circle subtend equal angles at the centre	Equal chords
If the angles subtended by the chords of a circle (or of congruent circles) at the centre are equal, then the chords are equal	Converse
2. Angle subtended by an arc of a circle	Angle subtended by arc
If two chords of a circle are equal, then their corresponding arcs are also equal	Equal arcs
The angle subtended by an arc at the centre is double the angle subtended by it at the remaining part of the circle	Angle at centre and circumference
The angle in a semi-circle is a right angle	Angle in semi-circle
Angles in the same segment of a circle are equal	Angles in same segment
3. Perpendicular from the centre to a chord	Perpendicular from centre to chord
The perpendicular from the centre of a circle to a chord bisects the chord	Perpendicular from centre to chord
The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord.	Converse
4. Perpendicular distance between the chord and the circle	Perpendicular distance to centre
The longer the chord, the nearer it is to the centre	Chord length vs centre
Equal chords of a circle are equidistant from the centre	Equal chords
Chords equidistant from the centre of a circle are equal	Converse
5. Circle through three non-collinear points	3 non-collinear points
A circle passing through three non-collinear points is	Unique circle

unique	
6. Concyclic points	
If a line segment joining two points subtends equal angle at two other points lying on the same side of the line containing the segment, the four points lie on a circle	Equal subtended angles
Cyclic quadrilaterals	
1. Cyclic quadrilaterials	
The sum of either pair of opposite angles of a cyclic quadrilateral is 180°	Sum of opposite angles
If the sum of any pair of opposite angles of a quadrilateral is 180°, then that quadrilateral is cyclic	Converse
Constructions	
1. Constructing bisectors of line segments	Bisectors of line segments
Constructing a bisectors of a line segment	
2. Constructing bisectors of angles	Bisectors of angles
3. Constructing angles of given measurements	Given angles
Constructing an angle of 90°	90^0
Constructing an angle of 60°	60^0
Constructing an angle of 45°	45^0
4. Constructing triangles	Triangles
Constructing an equilateral triangle	Equilateral triangle
Constructing a triangle given its base, the sum of the other two sides and one base angle	Given base, sum of 2 other sides, 1 base angle
Constructing a triangle given its base, the difference of the other two sides and one base angle	Given bae, difference of 2 other sides, 1 base angle
Constructing a triangle given its base, given its perimeter and base angles	Given base, perimeter, base angle

Basic terms

1. Basic terms

Parts of a circle

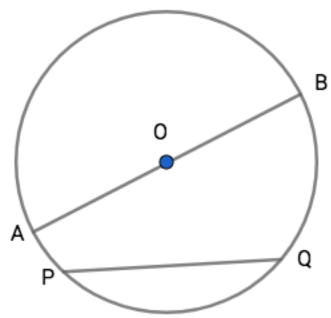
A **circle** is the collection of all points in a plane which are at an equal distance from a fixed point. The fixed point is called the **centre** and the distance between the centre and any point on the circle is called the **radius**.

We can say that a circle divides a plane into separate parts:

- The perimeter of a circle is called the **circumference**.
- The area inside the circle is called the **interior** of the circle.
- The area outside the circle is called the **exterior** of the circle.
- The circle and its interior together are called the **circular region**.

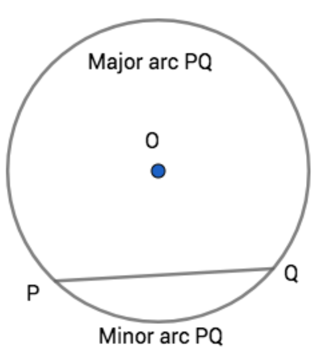
Lines and circles

Look at the diagram below.

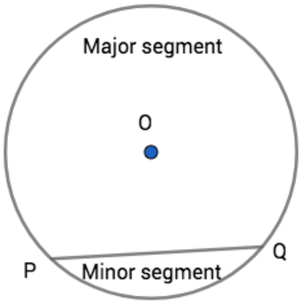


If you take any two points on the circle, say P and Q, the line segment PQ is called a **chord** of the circle. If the chord passes through the centre of the circle, it is called the **diameter** of the circle. In the diagram, AB is a diameter. The diameter of a circle is equal to two times the radius and it is the longest chord. All diameters of a given circle are equal in length.

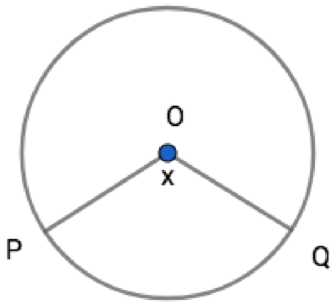
A section of a circle that lies between two points is called an **arc**. There are two sections of the circle that lie between the points P and Q on our diagram. The bigger section is called the **major arc** and the smaller section is called the **minor arc**.



The region between a chord and either of its arcs is called a **segment** of the circle. The bigger segment is called the major segment and the smaller segment is called the minor segment:



Now have a look at this diagram:

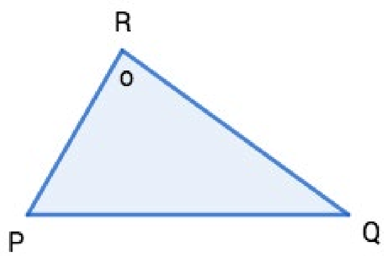


Here we see two radii, PO and QO, and an arc PQ. The region between the arc and the two radii is called a **sector**. The minor arc forms part of the **minor sector** and the major arc forms part of the **major sector**.

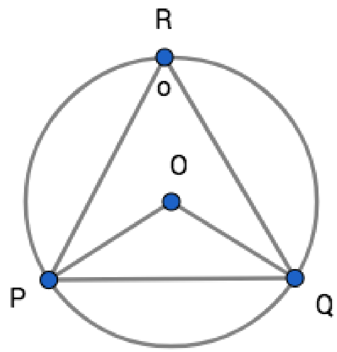
A diameter forms two arcs that are exactly equal, with the circle being divided into two **semi-circles**. In this case, both segments and both sectors will also be equal, and each of these is called a **semicircular region**.

Subtended angles

If we take a line segment PQ and a point R, which is not on the line containing PQ, and we join PR and QR, then the **angle subtended by line segment** PQ at point R is angle PRQ.



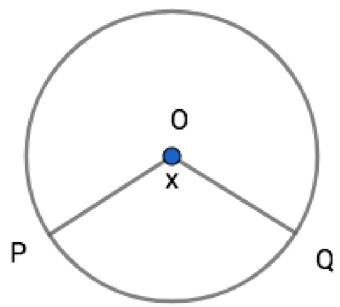
This is also used when working with circles. Look at the diagram below.



If we have a chord PQ, and a point R on the circle, then angle PRQ is the **angle subtended by chord** PQ at point R on the circle.

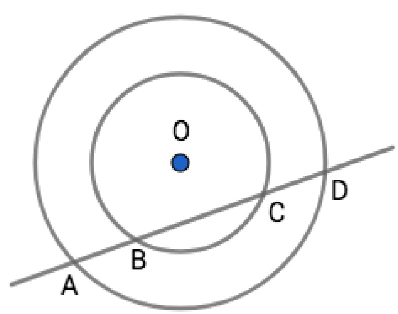
In the diagram below:

- The **angle subtended by minor arc PQ** at the centre of circle O, is angle POQ.
- The **angle subtended by major arc PQ** at the centre of the circle O, is the reflex angle POQ.

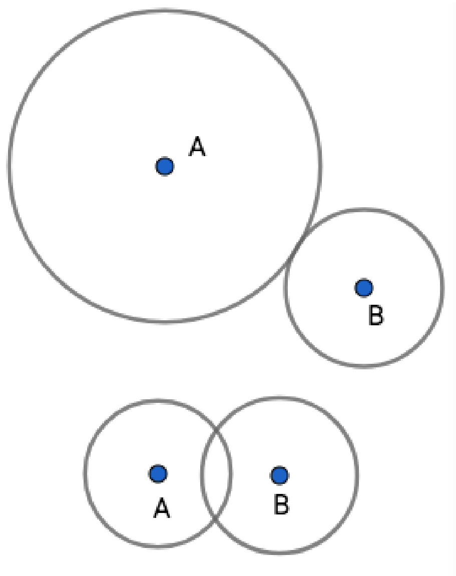


More than one circle

Concentric circles are circles which have the same centre:



Intersecting circles can intersect at one point or two points. If two circles are identical and lie directly on top of each other, they intersect at infinitely many points.



We say that two circles are **congruent** if they have the same radii.

Concyclic points are points that lie on the circumference of the same circle.

Circle theorems

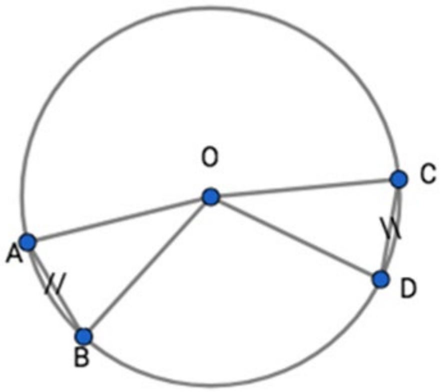
1. Chords & angles

Equal chords

Equal chords of a circle (or of congruent circles) subtend equal angles at the centre.

Proof:

Given: A circle with centre O:



Construct two equal chords, AB and CD. Join OA, OB, OC and OD.
Required to prove: $\angle AOB = \angle COD$

In $\triangle AOB$ and $\triangle COD$:

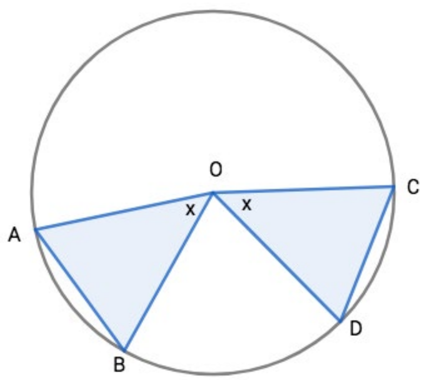
OA = OC	*radii of circle O*
OB = OD	*radii of circle O*
AB = CD	*given*
Therefore $\triangle AOB \equiv \triangle COD$	*SSS*
Therefore $\angle AOB = \angle COD$	*corresponding angles of congruent triangles*

Converse

If the angles subtended by the chords of a circle (or of congruent circles) at the centre are equal, then the chords are equal. This is the converse of the previous theorem.

Proof:

Given: A circle with centre O:



Construct two chords, AB and CD. Join OA, OB, OC and OD.
The chords are drawn such that the angles subtended by the chords at the centre, O, are equal: $\angle AOB = \angle COD$.
Required to prove: AB = CD

In $\triangle AOB$ and $\triangle COD$:

OA = OC	*radii of circle O*
OB = OD	*radii of circle O*
$\angle AOB = \angle COD$	*given*
Therefore $\triangle AOB \equiv \triangle COD$	*SAS*
So AB = CD	*corresponding sides of congruent triangles*

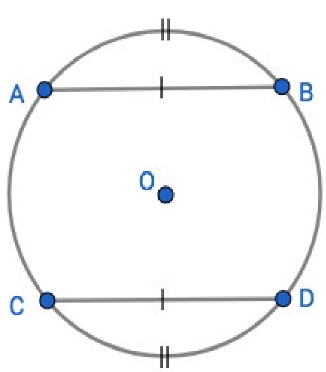
2. Angle subtended by arc

Equal arcs

If two chords of a circle are equal, then their corresponding arcs are also equal.

The converse of this is that if two arcs are equal, then their corresponding chords are also equal.

So in the diagram below, chords AB and CD are equal and arcs AB and CD are also equal.

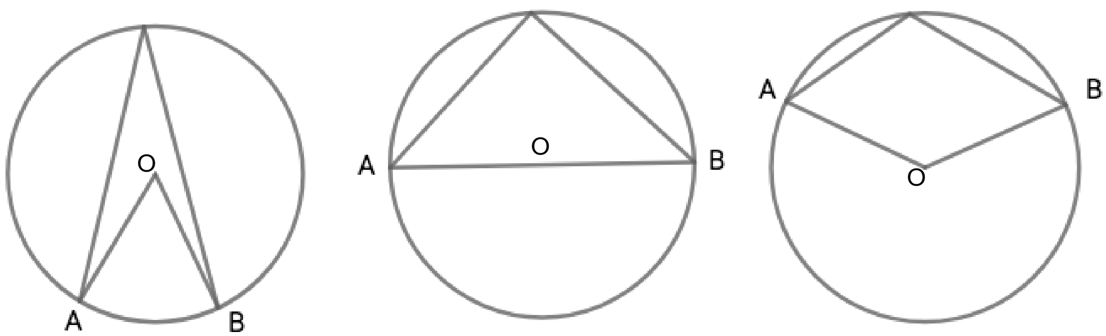


Angles at centre and circumference

The angle subtended by an arc at the centre is double the angle subtended by it at the remaining part of the circle.

Proof:

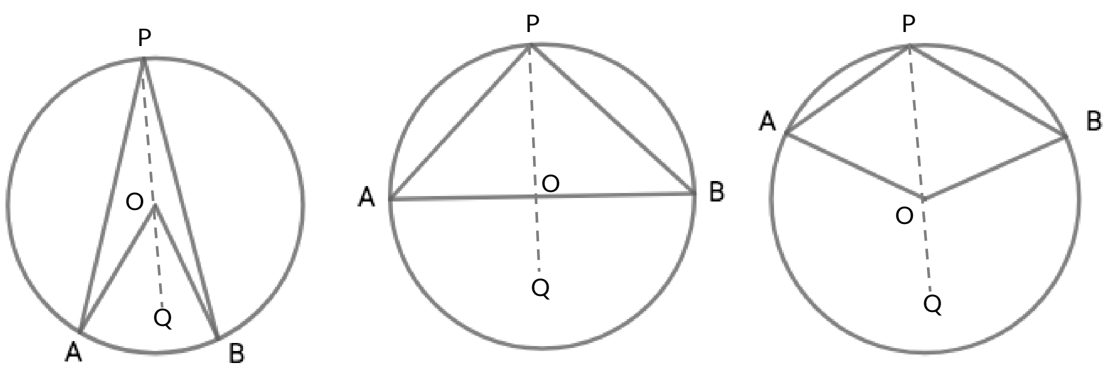
Given: A circle with centre O and arc AB. There are three types of arcs shown here: a minor arc, a semi-circle and a major arc.



Case 1: AB is a minor arc.
Case 2: AB is a semi-circle.
Case 3: AB is a major arc.

Required to prove: the angle subtended by an arc at the centre of the circle is twice the angle subtended at any other point on the circle. If P is any point on the circle, then we need to prove that ∠AOB = 2 × ∠APB.

Start by joining PO and extending it to Q:



In all cases:
In △AOP:

OA = OP	*radii*
So: ∠APO = ∠PAO	*angles opposite equal sides are equal*
∠AOQ = ∠APO + ∠PAO	*external angle equal to the sum of the two opposite internal angles*

Therefore ∠AOQ = ∠APO + ∠APO
So ∠AOQ = 2 × ∠APO

Similarly, $\angle BOQ = 2 \times \angle BPO$

So $\angle AOQ + \angle BOQ = 2 \times \angle APO + 2 \times \angle BPO$

$= 2 \times (\angle APO + \angle BPO)$

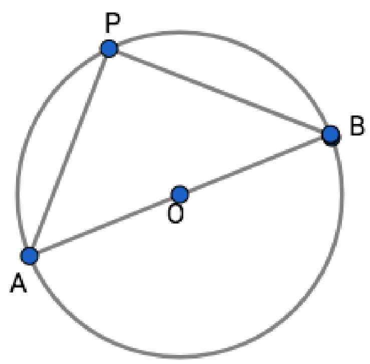
$= 2 \times \angle APB$

Angle in semi-circle

The angle in a semi-circle is a right angle.

We can use the previous theorem to show this.

We start with a circle with centre O and diameter AB. Take any point P on the circle and join AP and BP:



We know that:

$\angle AOB = 2 \times \angle APB$ *angle at centre is double angle subtended at the circle*

But $\angle AOB = 180°$, so:

$180° = 2 \times \angle APB$

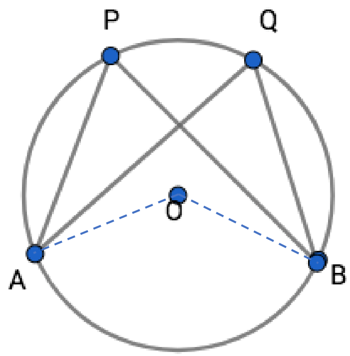
Therefore $\angle APB = 90°$

Angles in same segment

Angles in the same segment of a circle are equal.

Proof:

Given: A circle with centre O and arc AB:



Join OA and OB. Take any 2 points on the circle and label them P and Q. Then join AP, BP, AQ and BQ.
Required to prove: All angles in the segment subtended by AB are equal.

We know that $\angle APB = \frac{1}{2}\angle AOB$ *angle at centre is double angle subtended at the circle*

Similarly, $\angle AQB = \frac{1}{2}\angle AOB$ *angle at centre is double angle subtended at the circle*

So $\angle APB = \frac{1}{2}\angle AOB = \angle AQB$

Or: $\angle APB = \angle AQB$

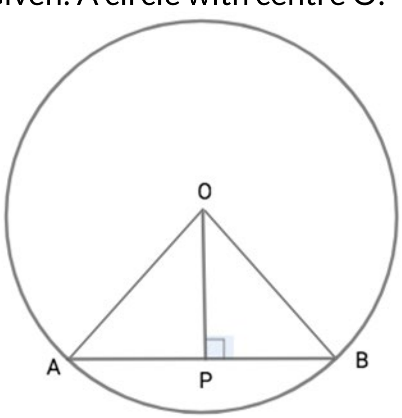
3. Perpendicular from centre to chord

Perpendicular from centre to chord

A perpendicular line drawn from a chord to the centre of the circle will bisect the chord.

Proof:

Given: A circle with centre O:



Construct chord AB. Then draw a line OP, which passes through centre O and intersects AB at P

such that OP $\perp$ AB. Join OA and OB.
Required to prove: AP = BP

In $\triangle$AOP and $\triangle$BOP:

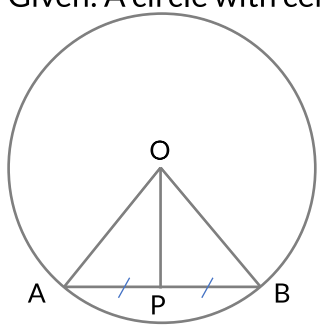
OA = OB	*radii of circle O*
OP = OP	*common side*
$\angle$OPA = $\angle$OPB = 90°	*given*
So $\triangle$AOP $\equiv$ $\triangle$BOP	*RHS*
Therefore AP = BP	*corresponding sides of congruent triangles*

Converse

The line drawn through the centre of a circle to bisect a chord is perpendicular to the chord. This is the converse of the previous theorem.

Proof:

Given: A circle with centre O:



Construct chord AB. Find the midpoint of AB, mark it P, and join PO. Join OA and OB.
Required to prove: OP $\perp$ AB

In $\triangle$AOP and $\triangle$BOP:

OA = OB	*radii of circle O*
OP = OP	*common side*
AP = BP	*P is the midpoint of AB*
So $\triangle$AOP $\equiv$ $\triangle$BOP	*SSS*
Therefore $\angle$OPA = $\angle$OPB	*corresponding angles of congruent triangles*

But:

$\angle$OPA + $\angle$OPB = 180° *linear pair of angles*

$\therefore$ $\angle$OPA + $\angle$OPA = 180°

$\therefore$ 2 × $\angle$OPA = 180°

$\therefore$ $\angle$OPA = $\angle$OPB = 90°

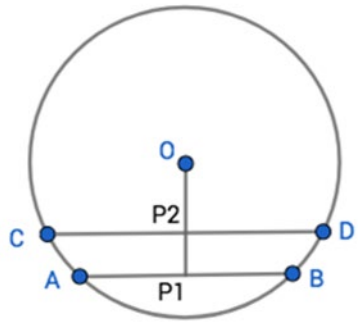
Therefore OP $\perp$ AB

4. Perpendicular distance to centre

Chord length vs centre

If two chords are drawn in the same circle, the longer chord will be closer to the centre of the circle.

Let's investigate this.
We start with a circle with centre O and a chord AB. Drop a perpendicular bisector from O, to meet AB at point P1. Now move closer to the centre of the circle and construct a new chord, CD:



In this diagram we can see that:

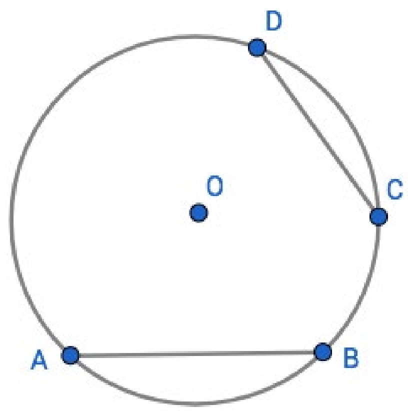
- The distance from the centre to CD is less than the distance from the centre to AB.
- CD > AB

Therefore the closer we move to point O, the greater the length of the chord. The longest possible chord would pass through point O.

Equal chords

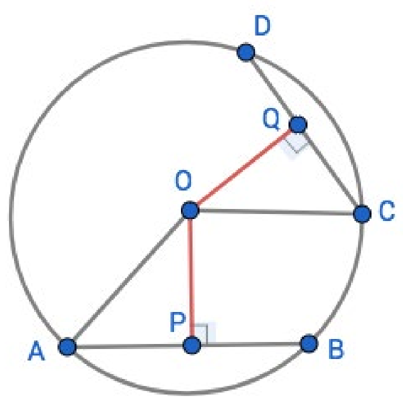
If two chords in the same circle are equal, then the chords are positioned at an equal distance from the centre of the circle.

Proof:
Given: A circle with centre O:



Construct any two equal chords, such as AB and CD.
Required to prove: Chords AB and CD are equidistant from point O.

Start by drawing a perpendicular from centre O to AB, intersecting at P. Then draw a perpendicular from centre O to CD, intersecting at Q. Join O and A, and join O and C:



Since OP $\perp$ AB, P bisects AB.

So: AP = $\frac{1}{2}$AB

Similarly, since OQ $\perp$ CD, Q bisects CD.

So: CQ = $\frac{1}{2}$CD

We also know that:

AB = CD *given*

$\therefore \frac{1}{2}$AB = $\frac{1}{2}$CD

Or: AP = CQ

In $\triangle$AOP and $\triangle$OCQ:

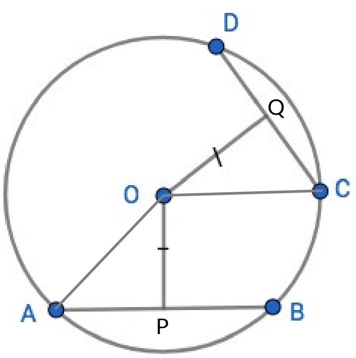
$\angle$OPA = $\angle$OQC *given; both angles = 90°*

OA = OC *radii of circle O*
AP = CQ *proven above*
So $\triangle AOP \equiv \triangle OCQ$ *RHS*
Therefore OP = OQ *corresponding sides of congruent triangles*
Therefore chords AB and CD are equidistant from centre O.

Converse

If two chords are the same distance from the centre of a circle, then these chords are equal in length. This is the converse of the previous theorem.

Proof:
Given: A circle with centre O:



Construct two chords, AB and CD, so that they are equidistant from centre O.
Construct a perpendicular line from centre O to AB, intersecting AB at P.
Similarly, construct a perpendicular line from centre O to CD, intersecting CD at Q.
The chords are equidistant from the centre, so OP = OQ. Join O and A, and join O and C.
Required to prove: AB = CD

Since OP $\perp$ AB, P bisects AB.
So: AP = $\frac{1}{2}$AB
Similarly, since OQ $\perp$ CD, Q bisects CD.
So: CQ = $\frac{1}{2}$CD

In $\triangle AOP$ and $\triangle OCQ$:
$\angle OPA = \angle OQC$ *given; both angles = 90°*
OA = OC *radii of circle O*
OP = OQ *given*
So $\triangle AOP \equiv \triangle OCQ$ *RHS*
Therefore AP = CQ *corresponding sides of congruent triangles*

So $\frac{1}{2}$AB = $\frac{1}{2}$CD *proven above*

$\therefore$ AB = CD

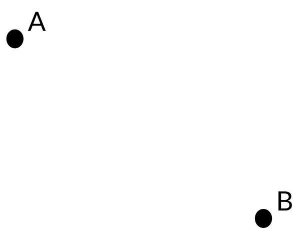
5. Three non-collinear points

Unique circle

There is only one unique circle that can be drawn to pass through three non-collinear points.

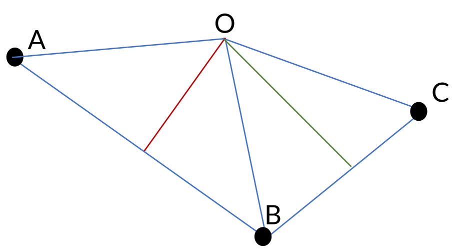
Proof:

Given: Three non-collinear points A, B and C:



Required to prove: There is only ONE unique circle that can pass through all three points (A, B and C).

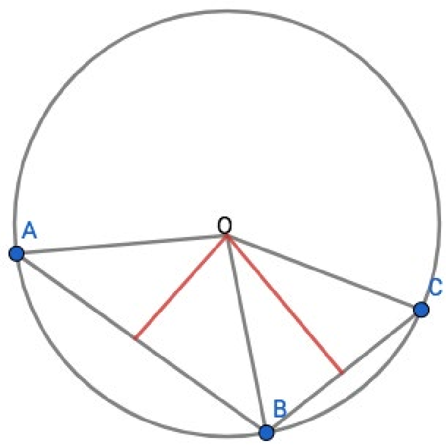
Join AB and BC. Then draw a perpendicular bisector of AB (shown in red below).
Also draw a perpendicular bisector of BC (shown in green below).
As AB and BC are not parallel, the perpendicular bisectors will not be parallel.
Call the point where the perpendicular bisectors intersect, O. Join AO, BO and CO.



Note that any point on the perpendicular bisector of AB (the red line) will be equidistant from both A and B.
O lies on this perpendicular bisector, so:
OA = OB
Similarly, any point on the perpendicular bisector of BC (the green line) will be equidistant from both C and B.
O also lies on this perpendicular bisector, so:
OC = OB
And therefore:
OA = OB = OC

We can see that point O is a point that is equidistant from all three points A, B and C.
This means that we can draw a circle with centre O and radius OA that will pass through B and C

(since OA = OB = OC):



As point O is a unique point, we can say that there is only ONE circle that passes through points A, B and C.

6. Concyclic points

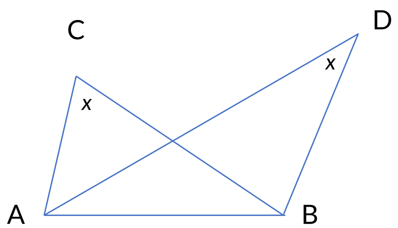
Concyclic points are points that lie on the circumference of the same circle.

Equal subtended angles

If a line segment joining two points subtends equal angles at two other points lying on the same side of the line containing the segment, the four points lie on a circle.

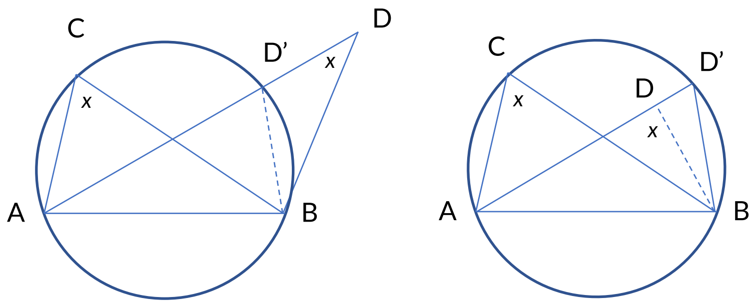
Proof:

Given: A line segment AB, with C and D two points that lie on the same side of the line that contains AB, and ∠ACB = ∠ADB.



Required to prove: A, B, C and D are concyclic.

Draw a circle through points A, B and C. Assume that point D does not lie on the circle, so the circle meets line AD at another point, which we can call D'. This could look like either of the following:



We now know that:

∠ACB = ∠ADB *given*

∠ACB = ∠AD'B *angles in the same segment are equal*

So ∠ADB = ∠AD'B

The only way that this can be true is if D and D' coincide so that D lies on the circle. Therefore points A, B, C and D must be concyclic.

Cyclic quadrilaterals

1. Cyclic quadrilaterals

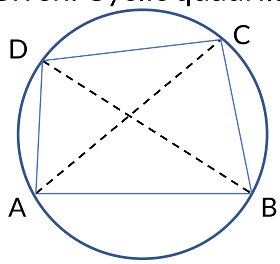
A **cyclic quadrilateral** is a quadrilateral with vertices that all touch the circumference of a circle.

Sum of opposite angles

The sum of either pair of opposite angles of a cyclic quadrilateral is 180°.

Proof:

Given: Cyclic quadrilateral ABCD with AC and BD joined:



Required to prove: $\angle A + \angle C = 180°$ and $\angle B + \angle D = 180°$

In the diagram:

$\angle ACB = \angle ADB$ *angles in same segment*

$\angle BAC = \angle BDC$ *angles in same segment*

Therefore $\angle ACB + \angle BAC = \angle ADB + \angle BDC = \angle D$

Add $\angle B$ to both sides of the equation:

$\angle ACB + \angle BAC + \angle B = \angle D + \angle B$

But $\angle ACB + \angle BAC + \angle B = 180°$ *sum of angles of a triangle = 180°*

So $180° = \angle D + \angle B$

We know that:

$\angle A+\angle B+\angle C+\angle D = 360°$ *sum of angles of a quadrilateral = 360°*

So if $\angle D + \angle B = 180°$

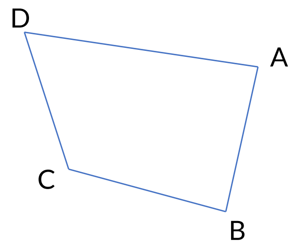
Then $\angle A + \angle C = 180°$

Converse

If the sum of any pair of opposite angles of a quadrilateral is 180°, then that quadrilateral is cyclic. This is the converse of the previous theorem.

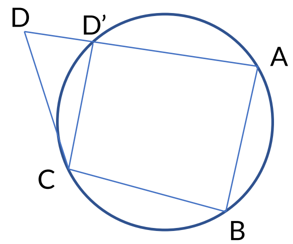
Proof:

Given: A parallelogram ABCD such that $\angle D + \angle B = 180°$:



Required to prove: ABCD is cyclic.

Draw a circle through points A, B and C. Let the circle meet AD at D':



ABCD' is a cyclic quadrilateral.
Therefore:
$\angle ABC + \angle AD'C = 180°$ *opposite angles of cyclic quadrilateral*
And $\angle ABC + \angle ADC = 180°$ *given*
$\therefore \angle ABC + \angle AD'C = \angle ABC + \angle ADC$
So $\angle AD'C = \angle ADC$

But $\angle AD'C$ is an external angle of triangle CD'D, and therefore cannot be equal to an external opposite angle.
Therefore the only way this can be valid is if D and D' coincide so that D lies on the circle.
This means that ABCD is a cyclic quadrilateral

Constructions

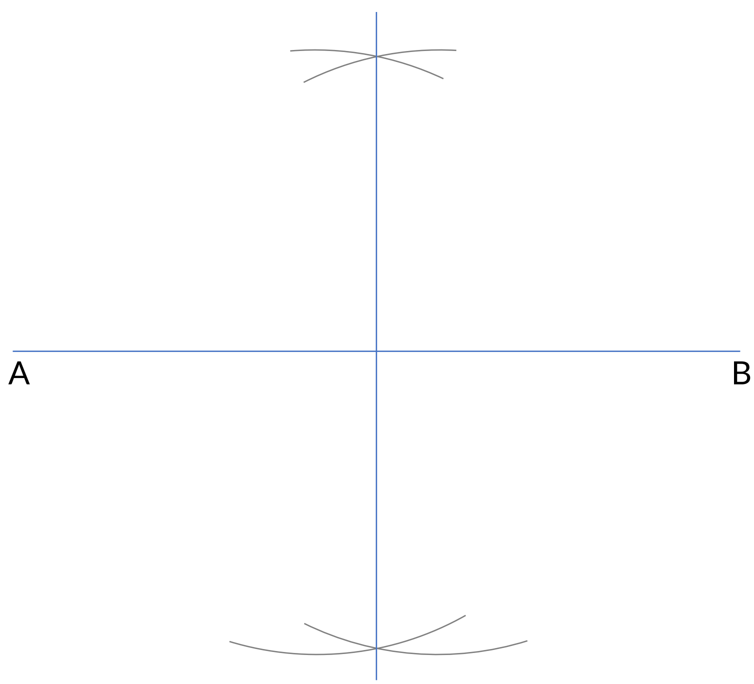
You can perform accurate constructions using a ruler, a sharp pencil and a pair of compasses.

1. Bisectors of line segments

Constructing a bisector of a line segment

To construct a line that bisects a given line segment:

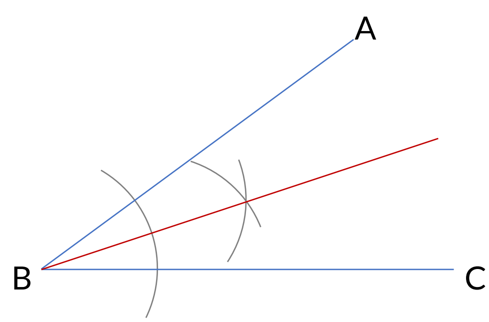
1. Start with the line segment that you need to bisect. Place the point of your compasses at the start point of the line (call this point A).
2. Open the compasses to a distance that is slightly longer than half the length of the line segment.
3. Keep the compasses at a fixed length and draw an arc above the line segment and another arc below the line segment.
4. With the compasses set to the same length, place the point on the end point of the line (call this point B).
5. Draw an arc above the line segment and another arc below the line segment, to intersect with the first set of arcs that you drew.
6. Use a ruler to draw a line through the points where the arcs intersect.



2. Bisectors of angles

To construct a line that bisects a given angle:

1. Start with the angle that you need to bisect. Label the angle ABC, with B at the vertex.
2. Set your compasses to a length that is about half the length of the arms of the angle. Place the point of your compasses at the vertex of the angle (point B).
3. Use the compasses to draw an arc that intersects with both the arms of the angle.
4. Keep the compasses set to exactly the same width. Move the point of the compasses to the point where the arc that you drew intersects the arm of the angle, AB.
5. Draw an arc in the space inside the angle, between AB and BC.
6. Keep the compass width the same and place the point where the original arc intersects with the other arm of the angle, BC.
7. Draw another arc in the space inside the angle, to intersect with the first.
8. Use a ruler to join the vertex with the point where the two arcs intersect.

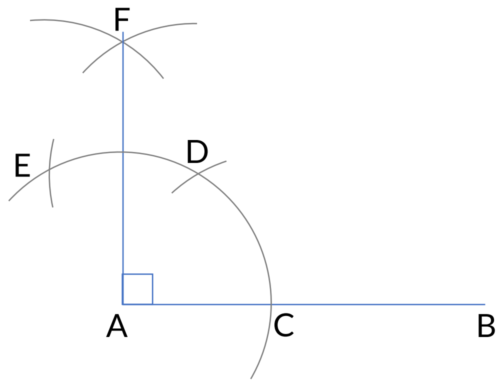


3. Given angles

90°

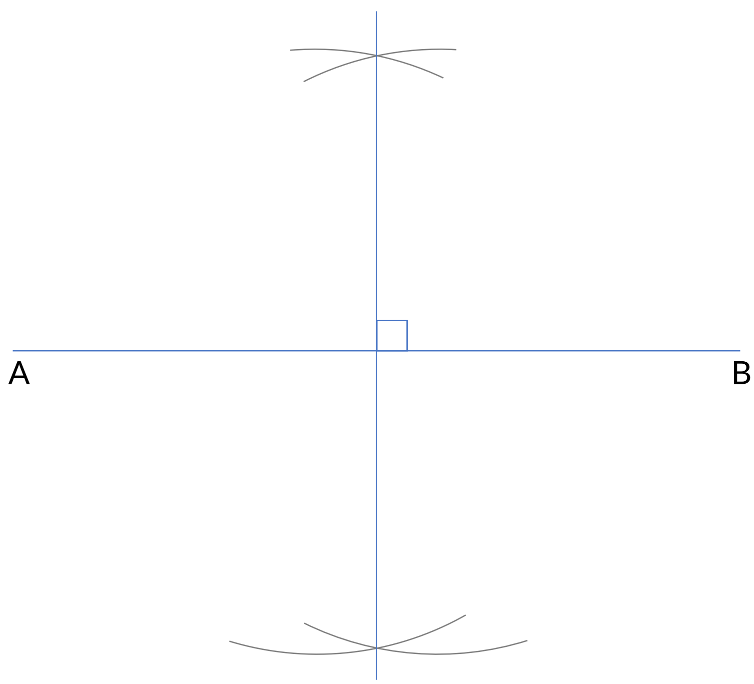
To construct an angle that is 90°:

1. Start by drawing a straight line and call this line AB.
2. Place your compass point at A and draw an arc that intersects line AB. Mark the point where the arc intersects with line AB and call it C.
3. Keep your compasses the same width and move the compass point to point C. Draw another arc that intersects with the first arc you drew. Call the point of intersection D.
4. Now keep the compasses the same width and move the point to point D. Draw another arc to intersect with your first arc, and label this point of intersection E.
5. Still keeping your compass at the same width, move the point first to D and then E, and in each case draw an arc in the direction opposite to the straight line AB. Mark the point where these lines intersect as F.
6. Use a ruler to join points F and A. This line will be perpendicular to line AB and will form an angle that is 90°.



You can also use this alternative method to draw an angle that is exactly 90°:

1. Follow steps 1 to 6 to construct a line bisector.
2. The bisector of a straight line will cut that line at 90°.

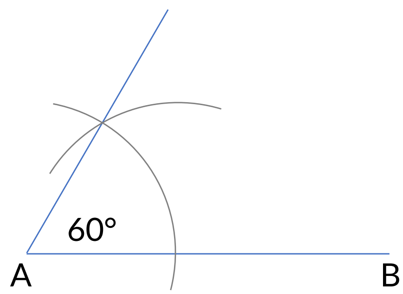


60°

To construct an angle that is 60°:

1. Start by drawing a straight line and label it AB.
2. Set your compasses to a suitable length and place the point on point A.
3. Draw an arc that intersects with the line AB and that extends above point A.
4. Keep the compass width the same and move the point to the place where the arc intersects with line AB.

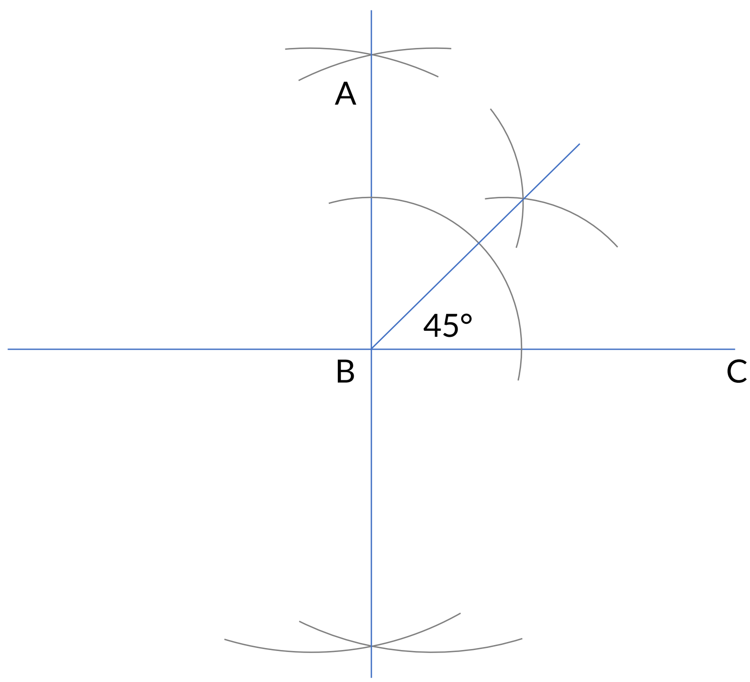
5. Draw another arc that intersects with the first one.
6. Use a ruler to draw a line that connects A with the intersection of the two arcs.



45°

To construct an angle that is 45°:

1. Follow the steps to draw a perpendicular bisector of a line AB, which will give you an angle of exactly 90°.
2. Follow the steps to bisect the angle, which will give you two angles that are each exactly 45°.



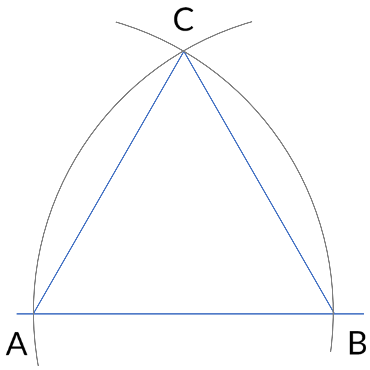
4. Triangles

Equilateral triangle

To construct an equilateral triangle:

1. Start by drawing a straight line. Mark a point on it and call this point A.
2. Set your compasses to the given length of one side of the equilateral triangle.

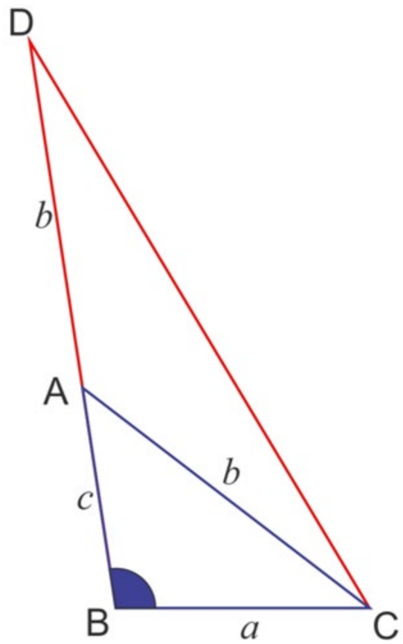
3. Use the compasses to draw an arc that passes through the straight line and extends above the line. Mark the point where the arc intersects the straight line as B.
4. Keep the compasses at the same width and move the point to point B.
5. Draw another arc that intersects the first arc and mark this point of intersection as C.
6. Use a ruler to connect points A, B and C with straight lines.



Given base, sum of two other sides, one base angle

If you are given the base length, the sum of the other two sides and a base angle, you can do the following:

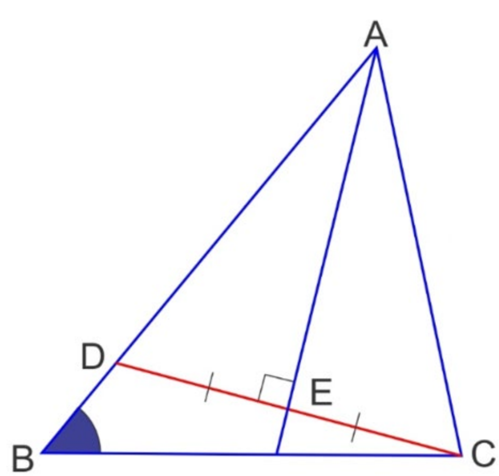
1. Start by constructing a line segment that is equal to the given length of the base. Call this line BC.
2. Measure the given base angle at point B and construct a line BD at this angle, which has a length that is equal to the sum of the other two sides of the triangle.
3. Join point C to point D.
4. Now construct a new line CA that will make triangle ACD an isosceles triangle (AD will be equal to AC).
5. The smaller triangle (triangle ABC) is the required triangle.



Given base, difference of two other sides, one base angle

If you are given the base length, the difference of the other two sides and a base angle, you can do the following:

1. Start by drawing a straight line that is equal to the given base length. Call this line BC.
2. Measure the given base angle at point B and construct a line BD at this angle, which has a length that is equal to the difference of the other two sides of the triangle.
3. Join point C to point D.
4. Construct a perpendicular bisector of line CD.
5. Extend the perpendicular bisector line and line BD until they meet. Call the point where they meet point A.
6. Join point A and point C. Triangle ABC is the required triangle.



Given base, perimeter, base angles

If you are given the perimeter of the triangle and its two base angles, you can do the following:

1. Start by drawing a straight line that is equal to the given perimeter.
2. Construct one of the base angles at one end of the line and the other base angle at the other end of the line.
3. Bisect the two angles that you have drawn, and let the bisectors intersect. Label the point where the bisectors intersect as point A.
4. Bisect the two lines that you have drawn as angle bisectors and extend the line bisectors so that they intersect with the original straight line that you drew.
5. Label the points where the line bisectors intersect with the original straight line, as B and C.
6. Join point A with points B and C. ABC is the required triangle.